Flag Algebras first try - Density approach, less formal

Flag algebras is a framework to study optimization problems over discrete structures. It is closely related to graph limits. It was introduced by Razborov in a paper from 2007. The framework can be applied to graphs, hypergraphs, oriented graphs, permutations, crossing numbers and many others. Today, we will focus only on graphs and we try to do more intuitive and less formal approach. More formal version will be next time.

In practice, flag algebras are useful for solving *optimization problems*, where both constraints and an objective function can be expressed by considering small subgraphs. For example Mantel's theorem:

$$\max\{|E(G)| : G \text{ is a } K_3\text{-free graph on } n \text{ vertices}\}.$$

The flag algebra results applies to *limits* and/or *large* graphs.

1: How to formulate the problem of maximizing the number of edges in triangle-free graphs as a limit problem? (i.e. $|V(G)| \to \infty$)

Solution:

$$\lim_{n\to\infty} \max\left\{|E(G)|/\binom{n}{2} : G \text{ is a } K_3\text{-free on } n \text{ vertices}\right\}$$

2: How can you get (lower) bounds on the limit problem above?

Solution: Find a good construction. In the example above, the complete bipartite graph will give a graph, where $|E(G)|/\binom{n}{2}$ tends to $1/2$. So we get a lower bound of $1/2$.

Flag algebras may help with the upper bound (but if lucky, it may help with the construction too).

Basic notation is for randomly picking a subgraph. For a graph G, we denote the number of vertices of G by $v(G)$.

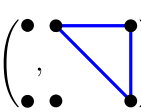
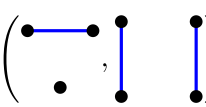
Definition: $P(H, G)$ is the probability that a set X of $v(H)$ vertices from $V(G)$ picked uniformly at random induces a copy of H.
In other words, pick $v(H)$ vertices from $V(G)$ randomly, denote the vertices X. $P(H, G)$ is the probability that $G[X]$ is isomorphic to H.
Of course, $v(G) \geq v(H)$.

Notice that $|E(G)|/\binom{n}{2}$ is the same as $P(K_2, G)$.

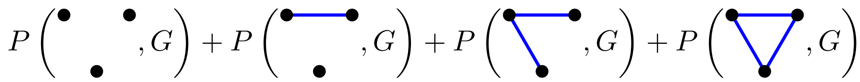
$$P(H,G) = \frac{\#\text{ of induced copies of } H \text{ in } G}{\binom{v(G)}{v(H)}}$$

In drawings,  is an edge and is a non-edge.

3: Calculate

$P\big($  $\big) = \frac{1}{2}$ $P\big($  $\big) = 1$

4: Let G be a graph on at least 3 vertices. Calculate

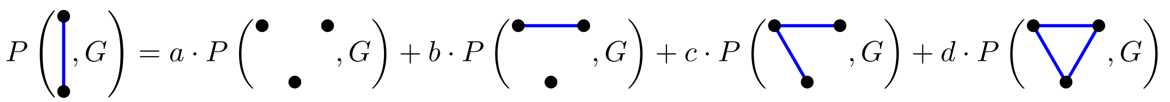


$$= 1$$

Solution: Since the sum contains ALL possible graphs on 3 vertices, the above events are disjoint and exactly one always happens.

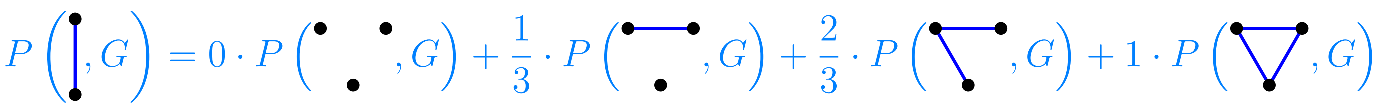
Our plan is to come up with similar equations as the one above. That is, expressions with small graphs that hold for ANY *large* graph G.

5: Find coefficients a, b, c, d independent of G such that the following equation is valid for any G.



(1)

Solution: The coefficients are $P(K_2, H)$. Think of the experiment as on the left hand side, pick randomly two vertices. On the right hand side, pick randomly three vertices and then pick two from these three.



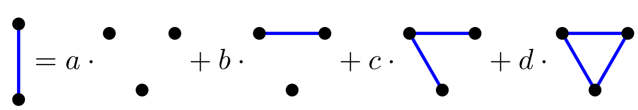
6: Let H be a graph, where $v(H) = k \leq \ell$. How to find c_{HF} in general such that

$$P(H, G) = \sum_{F, v(F)=\ell} c_{HF} \cdot P(F, G).$$

Solution:

$$c_{HF} = P(H, F)$$

Notice that in the previous question(s), we always write $P(?, G)$. It is somewhat inefficient writeup. So we now DROP all but ? from the notation and write just



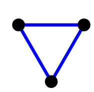
with the understanding, that every graph H in the expression actually means $P(H, G)$.

Our goal now is to give a slightly incorrect (in our notation) proof of Mantel's theorem.

7: Statement of Mantel's theorem is that if G is triangle free then

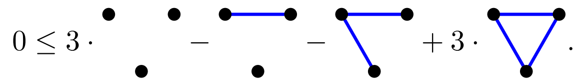
$$\leq \frac{1}{2} + o(1).$$

8: If G is triangle free, then



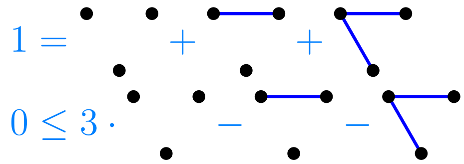
$$= 0.$$

9: Suppose that the following inequality is true for ANY G

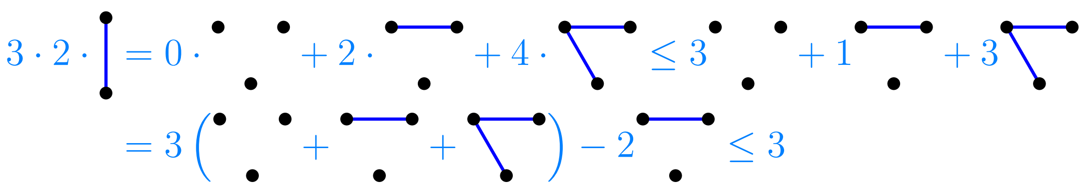
 (2)

This inequality is not quite true with the notation we have so far (Why?) but it will be true eventually for limits of large G. Can you combine it with the previous equations and inequalities to show the Mantel's theorem?

Solution: We use



Now

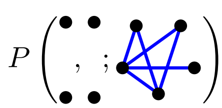


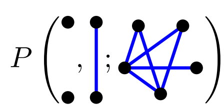
Now the question is, how to come up with inequalities like (2)? We will try to do it by using things like $0 \le (F_1 - F_2)^2$, which is true for any F_1 and F_2. To do so, we need to be able to somehow evaluate $F_1 \cdot F_2$. Recall it means $P(F_1, G) \cdot P(F_2, G)$. A key observation will be that if G is large, then insisting F_1 and F_2 are disjoint in $P(F_1, G) \cdot P(F_2, G)$ does not change the value too much.

Definition: $P(F_1, F_2; G)$, where F_1,F_2 and G are graphs with $v(G) \ge v(F_1) + v(F_2)$. Let X_1 and X_2 be two disjoint subsets of $V(G)$ of sizes $v(F_1)$ and $v(F_2)$ picked uniformly at random. The probability that $G[X_1]$ is isomorphic to F_1 and $G[X_2]$ is isomorphic to F_2 is denoted by $P(F_1, F_2; G)$.

Notice that the definition of $P(F_1, F_2; G)$ could be extended to $P(F_1, \ldots, F_k; G)$.

10: Calculate

 $= \frac{2}{30}$

 $= \frac{10}{30}$

11: Suppose G is a very large n-vertex graph compared to F_1 and F_2, where $v(F_1) = v(F_2) = \ell$ and ℓ is considered to be constant. Show that

$$|P(F_1, G) \cdot P(F_2, G) - P(F_1, F_2; G)| = O(n^{-1}).$$

Solution: Intuitively, in the product, if we pick randomly two ℓ-element subsets from an n-element set, where $n >> \ell$, we expect the two sets to be disjoint, so it is the same as $P(F_1, F_2; G)$. A little more formally. Let X_1 and X_2 be two subsets of $V(G)$ of size ℓ chosen uniformly at random independent of each other. Let A be the event that $G[X_1] \cong F_1$ and $G[X_2] \cong F_2$ and let B be the event that X_1 and X_2 are disjoint. Then

$$P(F_1, G) \cdot P(F_2, G) = P(A) \qquad P(F_1, F_2; G) = P(A|B).$$

Now

$$|P(A) - P(A|B)| \leq 1 - P(B) \leq P(X_1 \cap X_2 \neq \emptyset) \leq \frac{\ell\binom{n-1}{\ell-1}}{\binom{n}{\ell}} = \frac{l^2}{n} = O(n^{-1})$$

The first inequality follows from the following to cases:

$$\begin{aligned} P(A) - P(A|B) &\leq P(A) - P(A \wedge B) \cdot P(B)^{-1} \leq P(A \wedge B) + P(A \wedge \neg B) - P(A \wedge B) \\ &= P(A \wedge \neg B) \leq P(\neg B) = 1 - P(B) \\ P(A|B) - P(A) &\leq P(A|B) - P(A \wedge B) - P(A \wedge \neg B) \\ &= P(A|B) - P(A|B)P(B) - P(A \wedge \neg B) \leq P(A|B)(1 - P(B)) \leq 1 - P(B) \end{aligned}$$

12: Show that

$$P(F_1, F_2; G) = \sum_{H, v(H)=2\ell} P(F_1, F_2; H) \cdot P(H, G)$$

Solution: Law of total probability.

So now we have developed some kind of product that works as follow

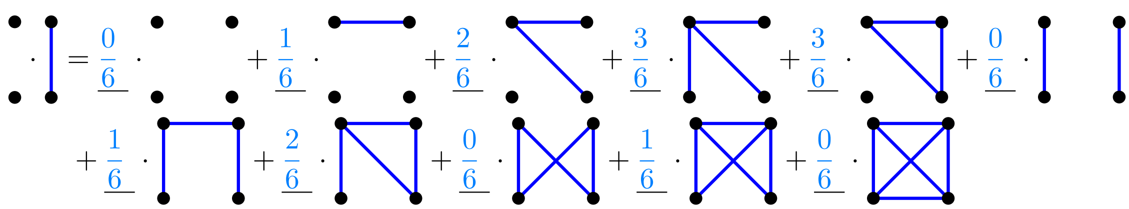
$$P(F_1, G) \cdot P(F_2, G) = P(F_1, F_2; G) + o(1) = o(1) + \sum_{H, v(H)=2\ell} P(F_1, F_2; H) \cdot P(H, G)$$

In our more compact notation, it can be written as

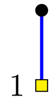
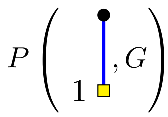
$$F_1 \cdot F_2 = \sum_{H, v(H)=2\ell} P(F_1, F_2; H) \cdot H + o(1)$$

We will also drop $o(1)$ in the expression that follow. But you should keep in mind that it is there. For example, (2) should contain $o(1)$ in the notation that we are currently using.

13: Fill in the missing coefficients in _.

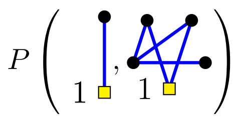
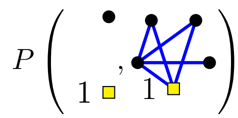


Next we introduce a concept that makes it easier to generate more inequalities like (2). Let σ be a (small) graph with vertices labeled by $1, \ldots, v(\sigma)$. This will be called *type*. Next everything we were doing so far, we can do again but EVERY graph we use must have a fixed labeled copy of σ and these copies must be mapped to each other when considering isomorphism. If we were picking something at random, pick only among unlabeled vertices and keep the labeled ones. Or alternatively the random pick must always contain the labeled vertices. We will denote the labeled vertices by squares ▫.

14: What does  mean if we use it as , where G also has one vertex labeled by 1?

Solution: Suppose G has n vertices. The labeled vertex gets mapped to the labeled vertex and then the unlabeled vertex is randomly chosen from the remaining $n-1$ unlabeled vertices in G. This corresponds to the degree of the labeled vertex divided by $n-1$.

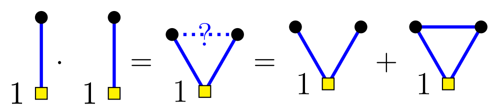
15: Calculate

 $= \frac{1}{2}$  $= \frac{1}{4}$

Now we try to do product but instead of doing

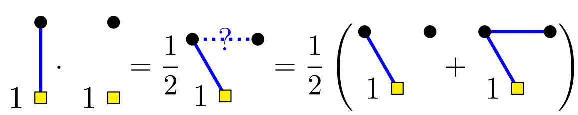
$$P(F_1, G) \cdot P(F_2, G) = P(F_1, F_2; G) = \sum_{H, v(H)=2\ell} P(F_1, F_2; H) \cdot P(H, G),$$

we try to do it differently. Let's do in on an example.



The first entry means pick randomly two (unlabeled) vertices with repetition and check that both are adjacent to the labeled vertex. The second asks pick randomly two distinct vertices and check that they are both adjacent to the labeled vertex; it does not matter how they are adjacent to each other.

16: If we try to multiply an edge and a non-edge, a extra factor $\frac{1}{2}$ has to be used. Why?

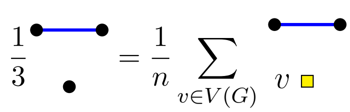
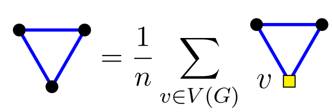


Solution: In the product on the left, we distinguish the first and the second choice but we do not do the distinction on the right. But it is a particular case. Best is always calculate $P(F_1, F_2; H)$ and use ? edges to generate the graphs H.

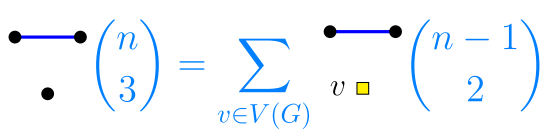
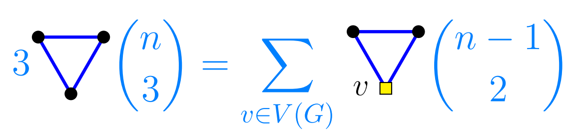
The above is to give some idea how the product might look like. Finally, the labeled graphs help with creating labeled equations. However, we are usually interested in unlabeled equations rather than labeled ones.

Lets try to do some small examples

17: To make the notation in the sum more clear, assume that the label vertex is v. Show that

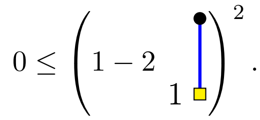
 and 

Solution: Just counting the number of $\overline{P}_3$ and K_3 in G.

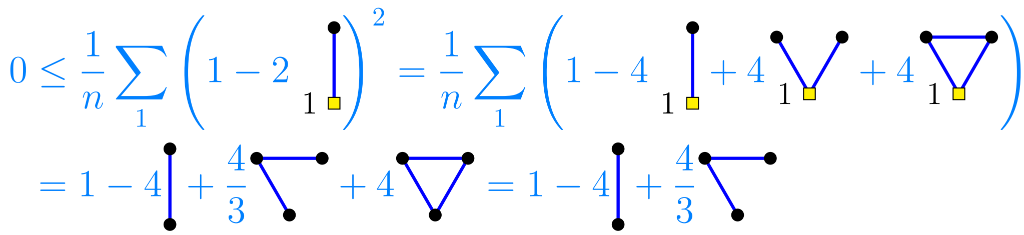
 

The operation above is called *averaging* or *unlabeling*. We will be more specific about the unlabeling next time.

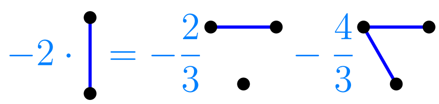
18: Prove again Mantel's theorem by using (1) and expanding and averaging the following



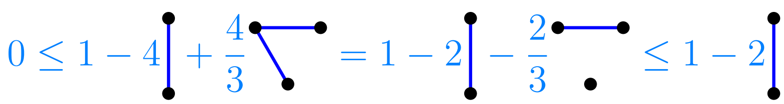
Solution:



By multiplying (1) by 2, we get

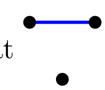


and continue the main calculation as

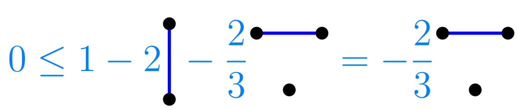


Hence  $\leq \frac{1}{2}$.

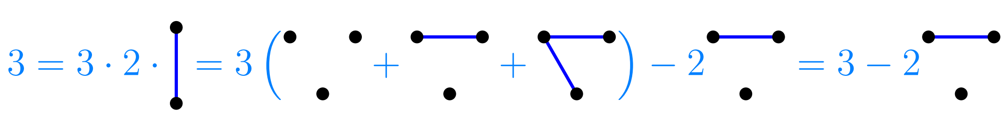
Finally we show how the calculations for Mantel's theorem give some information about the extremal example.

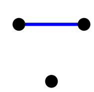
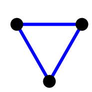
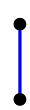
19: Suppose that  $= \frac{1}{2}$. Can you use any of the two previous calculations for  $\leq \frac{1}{2}$ to show that  $= 0$?

Solution: It is enough to use  $= \frac{1}{2}$ in

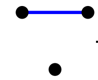
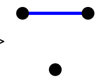


or



20: Suppose that a graph G does not contain ,  and  $= \frac{1}{2}$. Can you show it is a complete bipartite graph?

Solution: Take any edge. Every other vertex has exactly one neighbor and one non-neighbor on this edge. This creates a partition of all vertices into two sets. Try to look at the edges that remaining pairs and the forbidden triangles give that it must be a complete bipartite. Balancing is also possible.

Notice that flag algebras calculations above do not actually give that the extremal example is -free. Remember, there was $o(1)$ error in multiplication. The actual correct statement is that for every $\varepsilon > 0$, there exists n_0 such that every extremal graph on $n \geq n_0$ satisfies $\varepsilon >$ . Then there exists the following lemma, that makes the above valid up to εn^2 edges.

Lemma 1 (Infinite Removal Lemma, Alon and Shapira 2008)**.** *For any (possibly infinite) family $\mathcal{H}$ of graphs and $\varepsilon > 0$, there exists $\delta > 0$ such that if a graph G on n vertices contains at most $\delta n^{v(H)}$ induced copies of H for every graph H in $\mathcal{H}$, then it is possible to make G induced H-free, for every $H \in \mathcal{H}$, by adding and/or deleting at most εn^2 edges.*